SERBA: a B.I.E. program with linear elements for 2-D elastostatics analysis

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INTRODUCTION

The great developments that have occurred during the last few years in the finite element method and its applications has kept hidden other options for computation. The boundary integral element method now appears as a valid alternative and, in certain cases, has significant advantages. This method deals only with the boundary of the domain, while the F.E.M. analyses the whole domain. This has the following advantages: the dimensions of the problem to be studied are reduced by one, consequently simplifying the system of equations and preparation of input data. It is also possible to analyse infinite domains without discretization errors. These simplifications have the drawbacks of having to solve a full and non-symmetric matrix and some difficulties are incurred in the imposition of boundary conditions when complicated variations of the function over the boundary are assumed.

In this paper a practical treatment of these problems, in particular boundary conditions imposition, has been carried out using the computer program shown below.

Program SERBA solves general elastostatics problems in 2-dimensional continua using the boundary integral equation method.

The boundary of the domain is discretized by line or elements over which the functions are assumed to vary linearly.

Data (stresses and/or displacements) are introduced in the local co-ordinate system (element co-ordinates).

Resulting stresses are obtained in local co-ordinates and displacements in a general system.

The program has been written in Fortran ASCII and implemented on a 1108 Univac Computer. For 100 elements the core requirements are about 40 Kwords.

Also available is a Fortran IV version (3 segments) implemented on a 21 MX Hewlett-Packard computer, using 15 Kwords.

MATHEMATICAL MODEL

A large number of problems in mathematical physics may be expressed by the equation:

$$A \cdot u = f \tag{1}$$

where A is a linear operator or a set of them.

The solution of equation (1) can be put as:

$$u^* = \sum_{i=1}^{\infty} a_i \Psi_i \tag{2}$$

Projective methods study the solution of problem (1) using the following expression:

$$u^n = \sum_1^n a_i \Psi_i \tag{3}$$

If we define a space generated by φ_i and we project on it expression (1) substituting u from equation (3), we get:

$$\sum_{i=1}^{n} (A\Psi_i, \varphi_j) a_i = (f, \varphi_j) \tag{4}$$

Dealing with the problem using this equation, leads to a weak formulation of the original problem and to its resolution over the whole domain. As a particular case, when $\varphi_i = \Psi_i$, (functions are defined over all the domain), these are Rayleigh–Ritz methods and when using functions of small support we are looking at the F.E.M.

If we channel our study in order to get a boundary solution, it is only necessary to use the abstract Green's formula in expression (1) and the relation for φ:

$$\mathbf{A}\varphi = \mathbf{g} \tag{5}$$

In this way we obtain:

$$(u, A\varphi)_D = (A_u, \varphi)_D + (Eu, N\varphi)_{\partial D} - (E\varphi, Nu)_{\partial D} \tag{6}$$

where: D represents the domain under study, ∂D represents the boundary of the domain, N represents natural boundary conditions, E represents essential boundary conditions.

A suitable choice of φ allows simplification of expression (6) thus, when $A\varphi = 0$, we have Trefftz's methods and when $A\varphi = \delta(x_i)$ we have the B.I.E.M. in which equation (6) takes the form:

$$u(x_i) + (E\varphi, Nu)_{\partial D} = (Eu, N\varphi)_{\partial D} + (f, \varphi)_D \tag{7}$$

and if $x_i \in \partial D$:

$$c(x_i)u(x_i) + (E\varphi, Nu)_{\partial D} = (Eu, N\varphi)_{\partial D} + (f, \varphi)_D \tag{8}$$

we get an equation which only refers to the boundary, because the term extended to the domain does not include unknowns.

The advantage of carrying the problem to the boundary

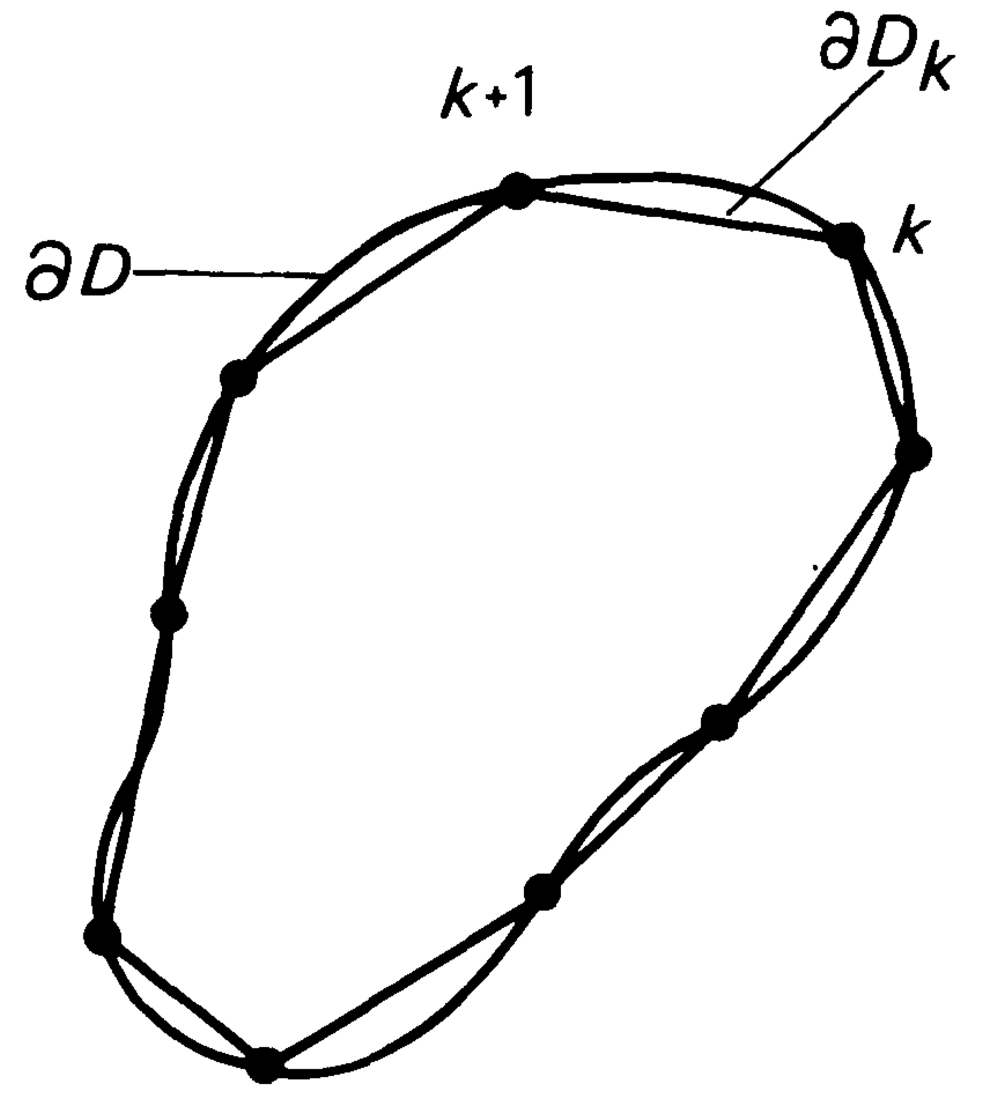

Figure 1.

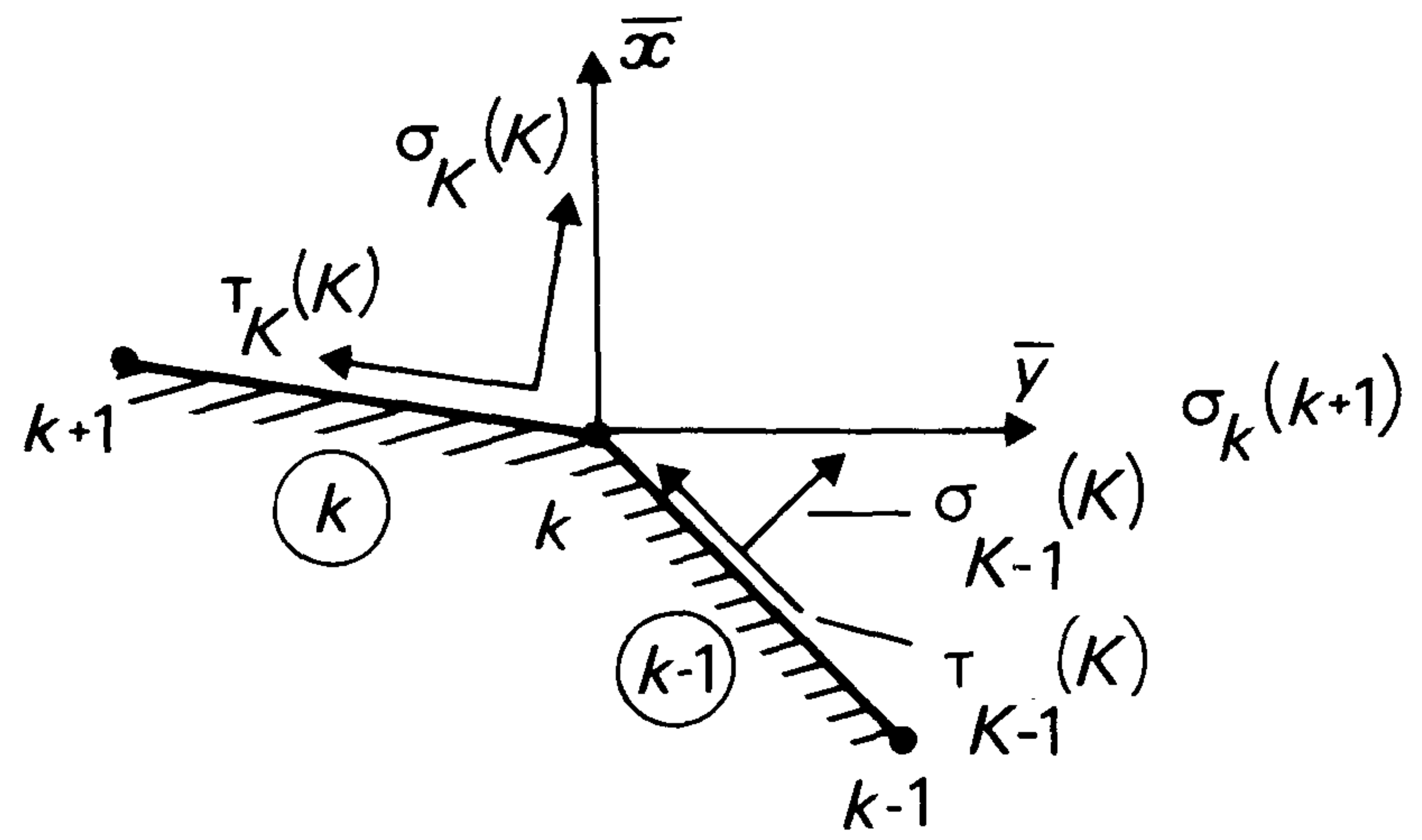

Figure 2. *Stresses in a node.* $\sigma_k(k+1) \equiv$ *normal stress at node* k + 1 *in the element* k

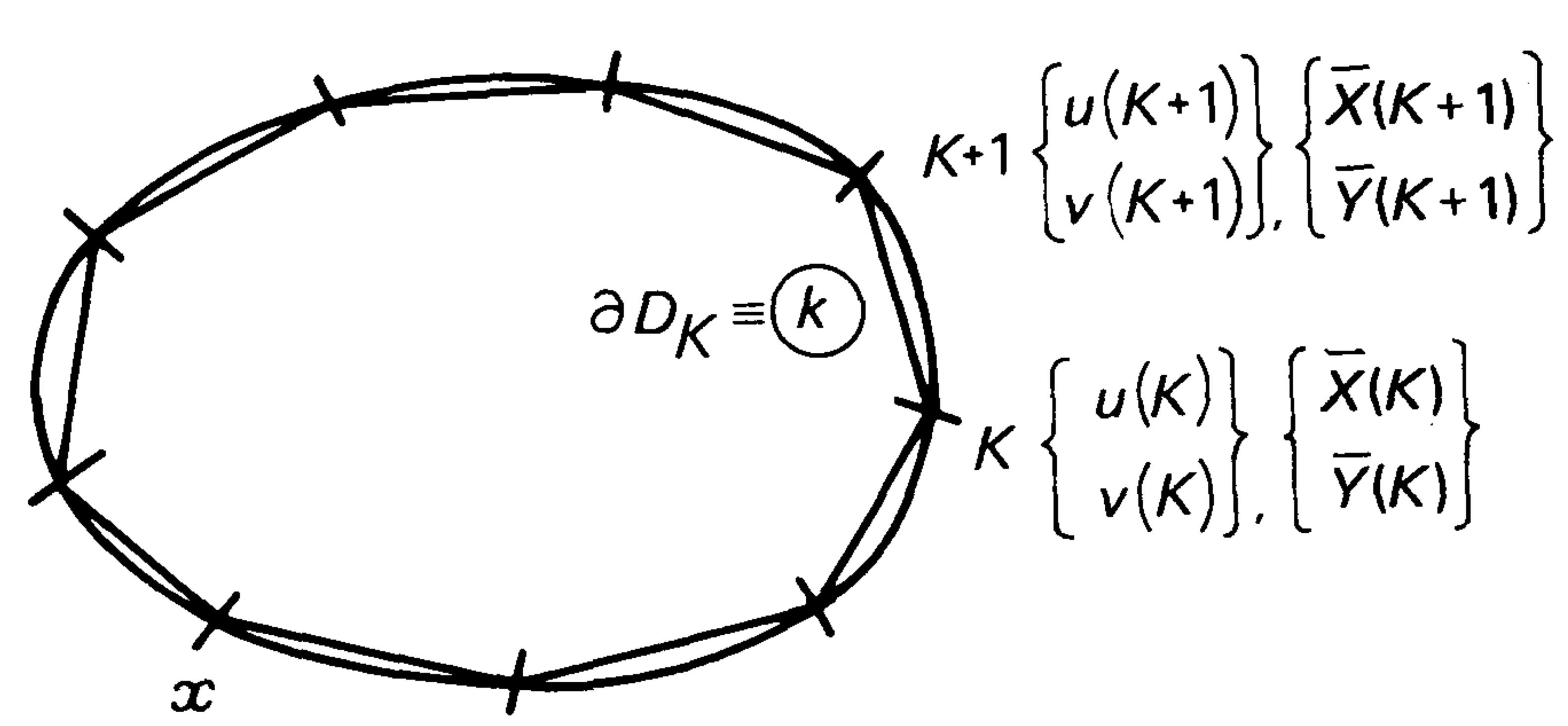

Figure 3.

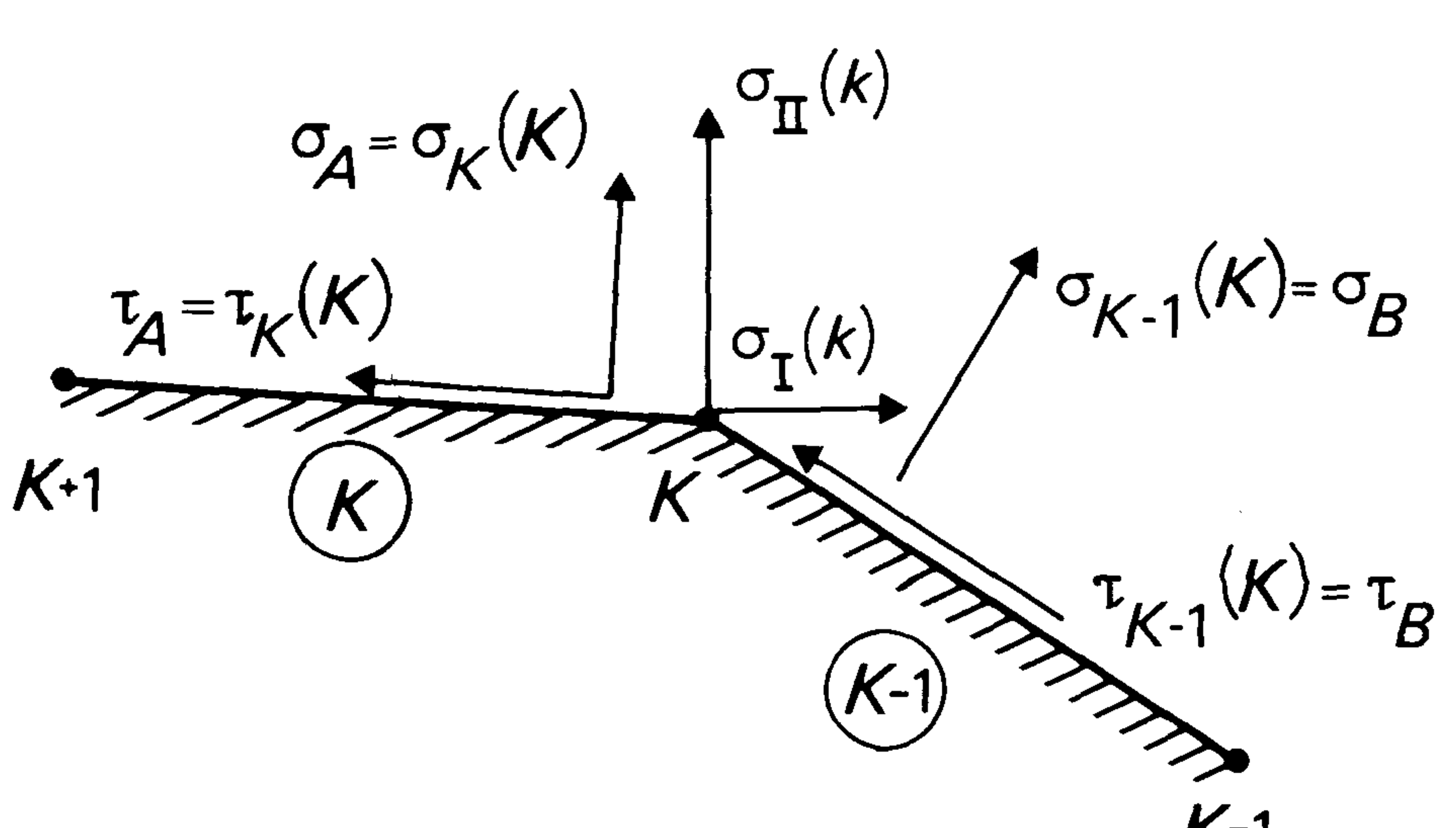

Figure 4.

has been paid for by losing the symmetry of the system which must now be expressed by means of a full matrix, since φ is defined over the whole boundary.

In elasticity problems, keeping in mind that $\varphi =$fundamental solution (i.e. Kelvin); $Nu = u \equiv$displacements; $Eu = t \equiv$stresses; $f =$body forces $(=0)$, expression (7) takes the form:

$$\mathbf{u} + \int_{\partial D} T \cdot \mathbf{u} \mathrm{d}s = \int_{\partial D} U t \mathrm{d}s \tag{9}$$

which is known as Somigliana's equation.

B.I.E.M. IMPLEMENTATION

Thus, the starting equation for using the method is equation (9) which can also be obtained through the use of a field equation (Navier equation), a reciprocity theorem (Maxwell–Betti) and a fundamental solution (Kelvin solution). In a more explicit form, expression (9) becomes:

$$u_j(x) + \int_{\partial D} T_{ji}(x, y) u_i(y) \mathrm{d}s = \int_{\partial D} U_{ji}(x, y) t_i(y) \mathrm{d}s \tag{10}$$

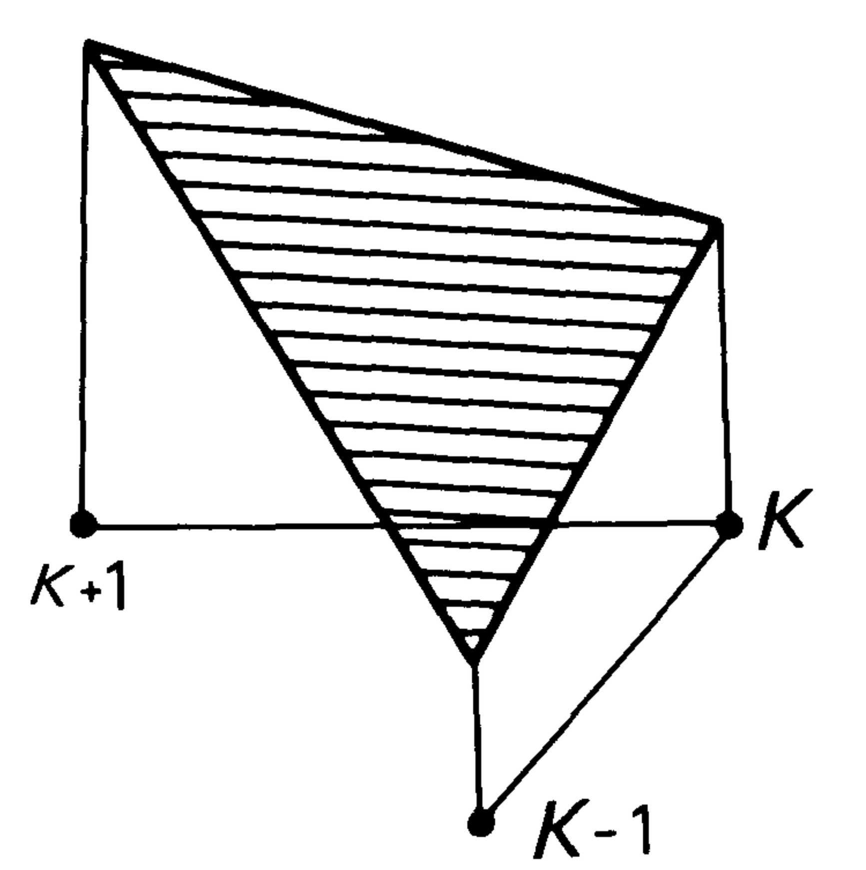

Figure 5.

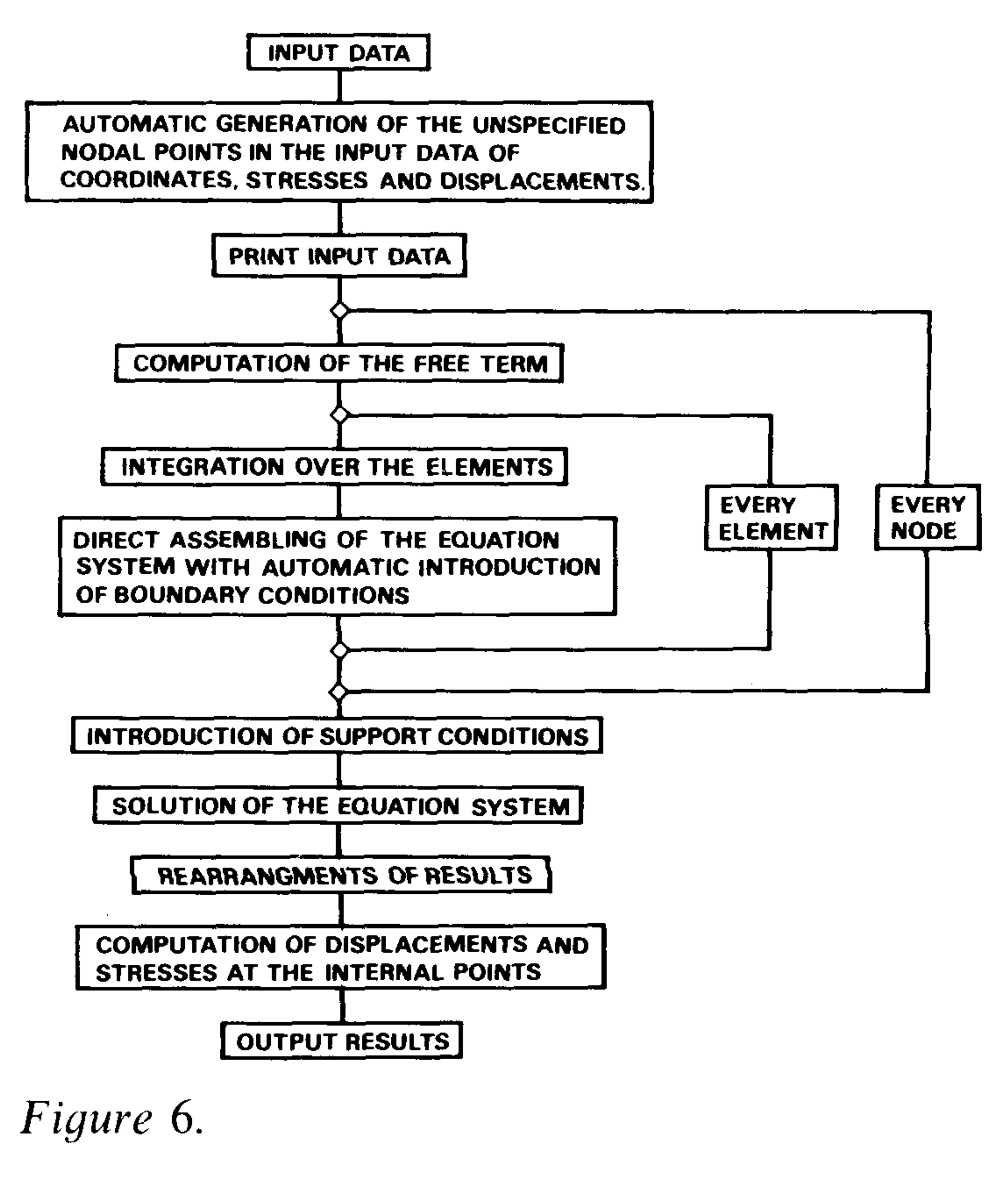

Figure 6.

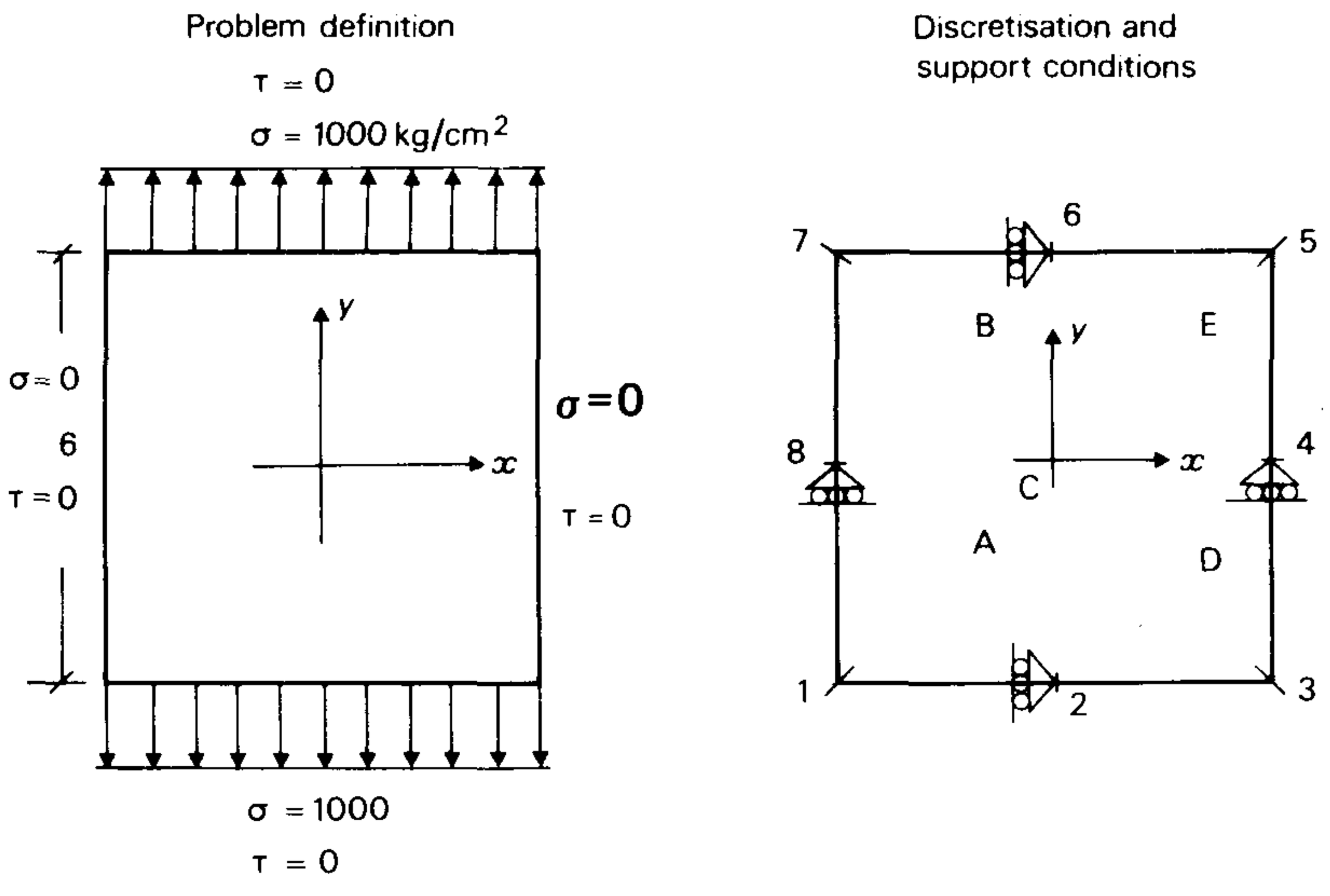

Figure 7. Example 1

```
 1        8    5    4             SQUARE PLATE
 2        1 2100000.  0.25          6.
 3     2.        2.
 4     2.        4.
 5     3.        3.
 6     4.        2.
 7     4.        4.
 8        1
 9        3 6.
10        5 6.         6.
11        7            6.
12        9
13        1    5               6              7 1000.     8
14        3    5 1000.         6              7           8
15        5    5               6              7  1000.    8
16        7    5 1000.         6              7           8
17        9    5               6              7 1000.     8
18        2    6   12   16
```

Figure 8. Input data cards for example 1

where x is any point of the domain, y is any point over the boundary, T_{ji}, U_{ji} are the Kelvin fundamental solution for the stresses and displacements, u, t represents the displacements and stresses in points of the domain or boundary. Equation (10) must be taken to the boundary and, to do so, it is necessary to make x tend to ∂D. In this case, the equation becomes:

$$[\delta_{ij}-C_{ij}(x)]u(x)+\int_{\partial D} T_{ji}(x,\ y)u_i(y)\mathrm{d}s=\int_{\partial D} U_{ji}(x,\ y)t_i(y)\mathrm{d}s \tag{11}$$

where both x and y are now over the boundary ∂D and:

$$C_{ij}(x)=\lim_{x\to y}\int_{\partial D(y)} T_{ji}(x,\ y)\mathrm{d}s$$

Expression (11) allows us to find values of displacements and stresses along the boundary of the domain. Then, equation (10) will let us find displacements at interior points and, finally, stresses at these points can be obtained from the displacements as follows:

$$\sigma_{ij}=\frac{2\mu\nu}{1-2\nu}\delta_{ij}\frac{\partial u_m}{\partial x_m}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{12}$$

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************************************************************************************************************
        PROGRAM SERBA FOR SOLUTION OF TWO DIMENSIONALELASTICITY PROBLEMS USING B.I.E.M.
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       INPUT DATA

       SQUARE PLATE

   NUMBER OF ELEMENTS =  8
   NUMBER OF INTERNAL POINTS =  5
   ELASTICITY MODULUS = 2100000.0
   POISSON RATIO =.200
   PROBLEM DEFINITION = 1
                  1 = PLANE STRESS
                  0 = PLANE STRAIN
```

NODE	COOR X	COOR Y	CODE	U	V	UG	VG	SGB	TAUB	SGA	TAUA
1	.0000	.0000	3456	.00000	.00000	.00000	.00000	.0000	.0000	.1000+004	.0000
2	3.0000	.0000	3456	.00000	.00000	.00000	.00000	.1000+004	.0000	.1000+004	.0000
3	6.0000	.0000	3456	.00000	.00000	.00000	.00000	.1000+004	.0000	.0000	.0000
4	6.0000	3.0000	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
5	6.0000	6.0000	3456	.00000	.00000	.00000	.00000	.0000	.0000	.1000+004	.0000
6	3.0000	6.0000	3456	.00000	.00000	.00000	.00000	.1000+004	.0000	.1000+004	.0000
7	.0000	6.0000	3456	.00000	.00000	.00000	.00000	.1000+004	.0000	.0000	.0000
8	.0000	3.0000	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
9	.0000	.0000	3456	.00000	.00000	.00000	.00000	.0000	.0000	.1000+004	.0000

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BOUNDARY RESULTS

NODE	XDISPLACEMENT	YDISPLACEMENT	SIGMA BEFORE	SIGMA AFTER	TAU BEFORE	TAU AFTER
1	.3570109-003	-.1401593-002	.0000000	.1000000+004	.0000000	.0000000
2	.0000000	-.1392953-002	.1000000+004	.1000000+004	.0000000	.0000000
3	-.3570114-003	-.1401592-002	.1000000+004	.0000000	.0000000	.0000000
4	-.3576201-003	.0000000	.0000000	.0000000	.0000000	.0000000
5	-.3570113-003	.1401592-002	.0000000	.1000000+004	.0000000	.0000000
6	.0000000	.1392953-002	.1000000+004	.1000000+004	.0000000	.0000000
7	.3570108-003	.1401593-002	.1000000+004	.0000000	.0000000	.0000000
8	.3576197-003	.0000000	.0000000	.0000000	.0000000	.0000000

INTERNAL POINTS RESULTS

COOR X	COOR Y	XDISPLACEMENT	YDISPLACEMENT	SIGMA X	SIGMA Y	TAU XY
2.000	2.000	.11980921-003	-.47291756-003	-.50738287+001	.99427104+003	-.11001481+001
2.000	4.000	.11980919-003	.47291760-003	-.50738049+001	.99427105+003	.11001513+001
3.000	3.000	-.18508217-009	.31377567-010	-.28037968+001	.99198519+003	.25033951-005
4.000	2.000	-.11980958-003	-.47291753-003	-.50738323+001	.99427100+003	.11001503+001
4.000	4.000	-.11980957-003	.47291758-003	-.50738056+001	.99427101+003	-.11001464+001

Figure 9. Results for example 1

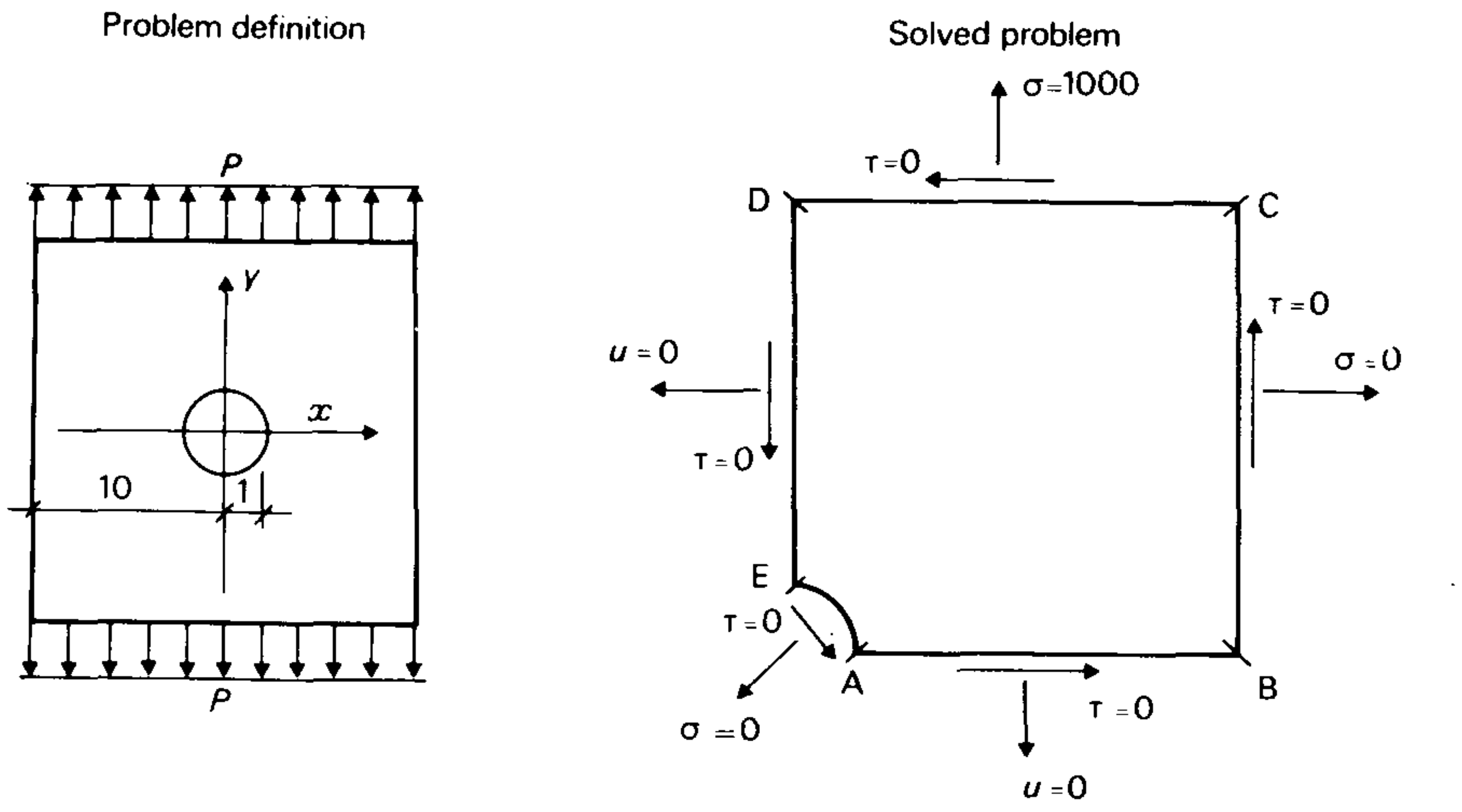

Figure 10. *Example 2: plate in traction with central hole*

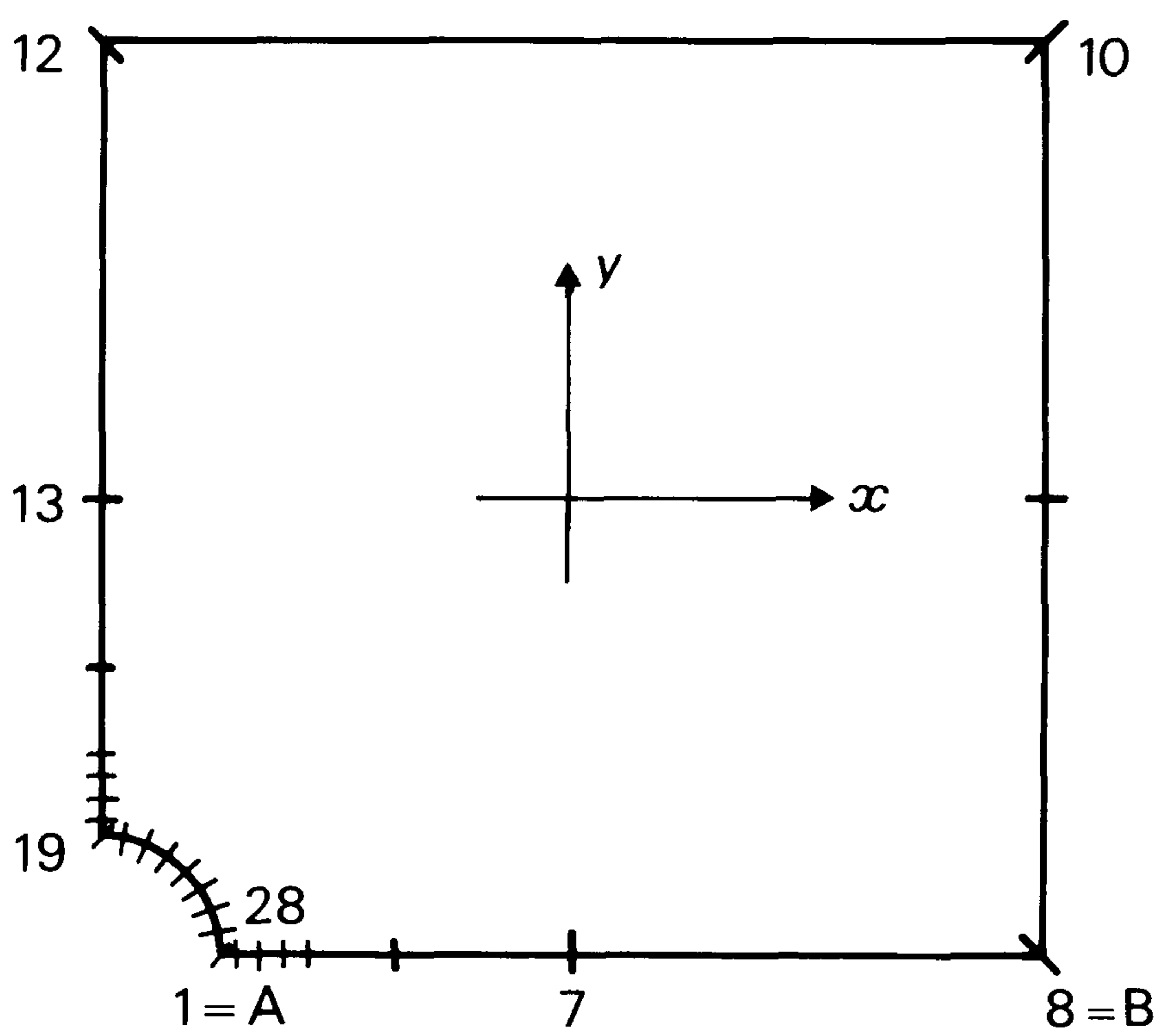

Figure 11. *Discretization used*

In order to discretize the integral equation, let's assume linear variation of functions (displacements and stresses) over the elements. In this way for an element ∂D_k:

$$\begin{Bmatrix} u \\ v \end{Bmatrix}_{\partial D_k} = \left(\begin{array}{c|c} N_1 N_2 & 0 \\ 0 & N_1 N_2 \end{array}\right) \begin{Bmatrix} u(K) \\ u(K+1) \\ v(K) \\ v(K+1) \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{X} \\ \bar{Y} \end{Bmatrix}_{\partial D_k} = \left(\begin{array}{c|c} N_1 N_2 & 0 \\ 0 & N_1 N_2 \end{array}\right) \begin{Bmatrix} \bar{X}(K) \\ \bar{X}(K+1) \\ \bar{Y}(K) \\ \bar{Y}(K+1) \end{Bmatrix} \tag{13}$$

where:

$$N_1 = -\tfrac{1}{2}(\zeta - 1) \qquad 1 \leqslant \zeta \leqslant 1$$

$$N_2 = \tfrac{1}{2}(\zeta + 1)$$

$u(k)$ = displacements along x axis at node k, $v(k+1)$ = displacements along y axis at node $k+1$, $\bar{X}(k)$ = stress in x direction at node k, $\bar{Y}(k+1)$ = stress in $\bar{Y}$ direction at node $k+1$.

It is usual to refer stresses (data and/or unknowns) to local coordinates,

$$\begin{Bmatrix} \bar{X}(k) \\ \bar{X}(k+j) \\ \bar{Y}(k) \\ \bar{Y}(k+j) \end{Bmatrix} = \mathbf{M} \begin{Bmatrix} \sigma_k(k) \\ \sigma_k(k+1) \\ \tau_k(k) \\ \tau_k(k+1) \end{Bmatrix} \tag{14}$$

where **M** is a transformation matrix which depends on the unit vector that define the element orientation, and the rotation for the local stresses.

Therefore the integral equation applied to a node x, and referring it to an element k placed between nodes k and $k+1$ is:

$$\begin{pmatrix} 1-C_{11} & -C_{12} \\ -C_{12} & 1-C_{22} \end{pmatrix} \begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} + \sum_{k=1}^{N} \int_{\partial D_k} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}_{\partial D_k} \mathrm{d}s_k$$

$$= \sum_{k=1}^{N} \int_{\partial D_k} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{Bmatrix} \bar{X} \\ \bar{Y} \end{Bmatrix}_{\partial D_k} \mathrm{d}s_k \tag{15}$$

Making use of equations for displacements and stresses given in equation (13) and according to the modification (14), expression (15), after some operations, takes the form:

$$\begin{pmatrix} 1-C_{11} & -C_{12} \\ -C_{21} & 1-C_{22} \end{pmatrix} \begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} + \sum_{k=1}^{N} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \end{bmatrix} \times$$

$$\begin{Bmatrix} u(k) \\ u(k+1) \\ v(k) \\ v(k+1) \end{Bmatrix} =$$

$$= \sum_{k=1}^{N} \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \end{bmatrix} \mathbf{M} \begin{Bmatrix} \sigma_k(k) \\ \sigma_k(k+1) \\ \tau_k(k) \\ \tau_k(k+1) \end{Bmatrix} \tag{16}$$

```
 1        28    9                   PLATE IN TRACTION WITH CENTRAL HOLE
 2         1 2100000.   0.25          10.
 3      1.         5.
 4      3.         5.
 5      5.         5.
 6      7.         5.
 7      9.         5.
 8      5.         1.
 9      5.         3.
10      5.         7.
11      5.         9.
12         1 1.
13         3 1.2
14         4 1.4
15         6 2.
16         7 3.
17         8 5.
18         9 10.
19        11 10.        10.
20        13            10.
21        14            5.
22        15            3.
23        16            2.
24        18            1.4
25        19             1.2
26        21             1.
27        22.098        .9951
28        23.1950       .9807
29        24.3826       .9238
30        25.7071       .7071
31        26.9238       .3826
32        27.9807       .1950
33        28.9951       .0980
34         1     3                5            6            8
35         9     1                6            7            8
36        11     5                6            7 1000.      8
37        13     3                5 1000.      6            8
38        21     1                6            7            8
39        29     3                5            6            8
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Figure 12. *Input data cards for example 2*

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************************************************************************************************************************
                    PROGRAM SERBA FOR SOLUTION OF TWO DIMENSIONALELASTICITY PROBLEMS USING B.I.E.M.
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        INPUT DATA


        PLATE IN TRACTION WITH CENTRAL HOLE

    NUMBER OF ELEMENTS = 28
    NUMBER OF INTERNAL POINTS =  9
    ELASTICITY MODULUS = 2100000.0
    POISSON RATIO =.200
    PROBLEM DEFINITION = 1
                    1 = PLANE STRESS
                    0 = PLANE STRAIN
```

NODE	COOR X	COOR Y	CODE	U	V	UG	VG	SGB	TAUB	SGA	TAUA
1	1.0000	.0000	1346	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
2	1.1000	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
3	1.2000	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
4	1.4000	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
5	1.7000	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
6	2.0000	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
7	3.0000	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
8	5.0000	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
9	10.0000	.0000	1456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
10	10.0000	5.0000	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
11	10.0000	10.0000	3456	.00000	.00000	.00000	.00000	.0000	.0000	.1000+004	.0000
12	5.0000	10.0000	3456	.00000	.00000	.00000	.00000	.1000+004	.0000	.1000+004	.0000
13	.0000	10.0000	1346	.00000	.00000	.00000	.00000	.1000+004	.0000	.0000	.0000
14	.0000	5.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
15	.0000	3.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
16	.0000	2.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
17	.0000	1.7000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
18	.0000	1.4000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
19	.0000	1.2000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
20	.0000	1.1000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
21	.0000	1.0000	1456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
22	.0980	.9951	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
23	.1950	.9807	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
24	.3826	.9238	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
25	.7071	.7071	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
26	.9238	.3826	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
27	.9807	.1950	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
28	.9951	.0980	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
29	1.0000	.0000	1346	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000

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BOUNDARY RESULTS

NODE	XDISPLACEMENT	YDISPLACEMENT	SIGMA BEFORE	SIGMA AFTER	TAU BEFORE	TAU AFTER
1	-.5059891-003	.0000000	.0000000	.3225943+004	.0000000	.0000000
2	-.5319561-003	.0000000	.2440763+004	.2440763+004	.0000000	.0000000
3	-.5446807-003	.0000000	.2067366+004	.2067366+004	.0000000	.0000000
4	-.5524092-003	.0000000	.1632551+004	.1632551+004	.0000000	.0000000
5	-.5525474-003	.0000000	.1312960+004	.1312960+004	.0000000	.0000000
6	-.5545430-003	.0000000	.1217049+004	.1217049+004	.0000000	.0000000
7	-.5966076-003	.0000000	.1067579+004	.1067579+004	.0000000	.0000000
8	-.7699718-003	.0000000	.1047604+004	.1047604+004	.0000000	.0000000
9	-.1332364-002	.0000000	.9783081+003	.0000000	.0000000	.0000000
10	-.1249260-002	.2311125-002	.0000000	.0000000	.0000000	.0000000
11	-.1103682-002	.4654054-002	.0000000	.1000000+004	.0000000	.0000000
12	-.5113579-003	.4795909-002	.1000000+004	.1000000+004	.0000000	.0000000
13	.0000000	.4930943-002	.1000000+004	.6195603+002	.0000000	.0000000
14	.0000000	.2664324-002	.3511486+001	.3511486+001	.0000000	.0000000
15	.0000000	.1852894-002	.2655658+002	.2655658+002	.0000000	.0000000
16	.0000000	.1550839-002	.8597320+001	.8597320+001	.0000000	.0000000
17	.0000000	.1493032-002	-.9054823+001	-.9054823+001	.0000000	.0000000
18	.0000000	.1459127-002	-.1399525+003	-.1399525+003	.0000000	.0000000
19	.0000000	.1449738-002	-.3942443+003	-.3942443+003	.0000000	.0000000
20	.0000000	.1446151-002	-.6271700+003	-.6271700+003	.0000000	.0000000
21	.0000000	.1439057-002	-.1189019+004	.0000000	.0000000	.0000000
22	-.5022593-004	.1431237-002	.0000000	.0000000	.0000000	.0000000
23	-.1017680-003	.1409361-002	.0000000	.0000000	.0000000	.0000000
24	-.1944866-003	.1321321-002	.0000000	.0000000	.0000000	.0000000
25	-.3443741-003	.1006785-002	.0000000	.0000000	.0000000	.0000000
26	-.4616130-003	.5616454-003	.0000000	.0000000	.0000000	.0000000
27	-.4950339-003	.2882587-003	.0000000	.0000000	.0000000	.0000000
28	-.5025733-003	.1422806-003	.0000000	.0000000	.0000000	.0000000

INTERNAL POINTS RESULTS

COOR X	COOR Y	XDISPLACEMENT	YDISPLACEMENT	SIGMA X	SIGMA Y	TAU XY
1.000	5.000	-.10164673-003	.26330696-002	.72475887+002	.85440762+003	.52565670+002
3.000	5.000	-.35265159-003	.25367631-002	-.26275937+002	.10053657+004	-.41632241+002
5.000	5.000	-.62058478-003	.24417726-002	-.23346517+002	.10232934+004	-.12213824+002
7.000	5.000	-.87945210-003	.23811900-002	-.10412292+002	.10105771+004	.29810838+001
9.000	5.000	-.11551590-002	.23127079-002	-.15904837+003	.10746218+004	-.14104422+003
5.000	1.000	-.75317669-003	.45641438-003	.46709703+002	.10262772+004	-.74515846+001
5.000	3.000	-.68039894-003	.14512628-002	-.19344577+002	.10425967+004	.87306861+001
5.000	7.000	-.58112047-003	.34098175-002	-.41083089+001	.10007872+004	-.15917080+002
5.000	9.000	-.53521150-003	.44501775-002	-.30382431+003	.15333404+004	.24517436+002

Figure 13. Results for example 2

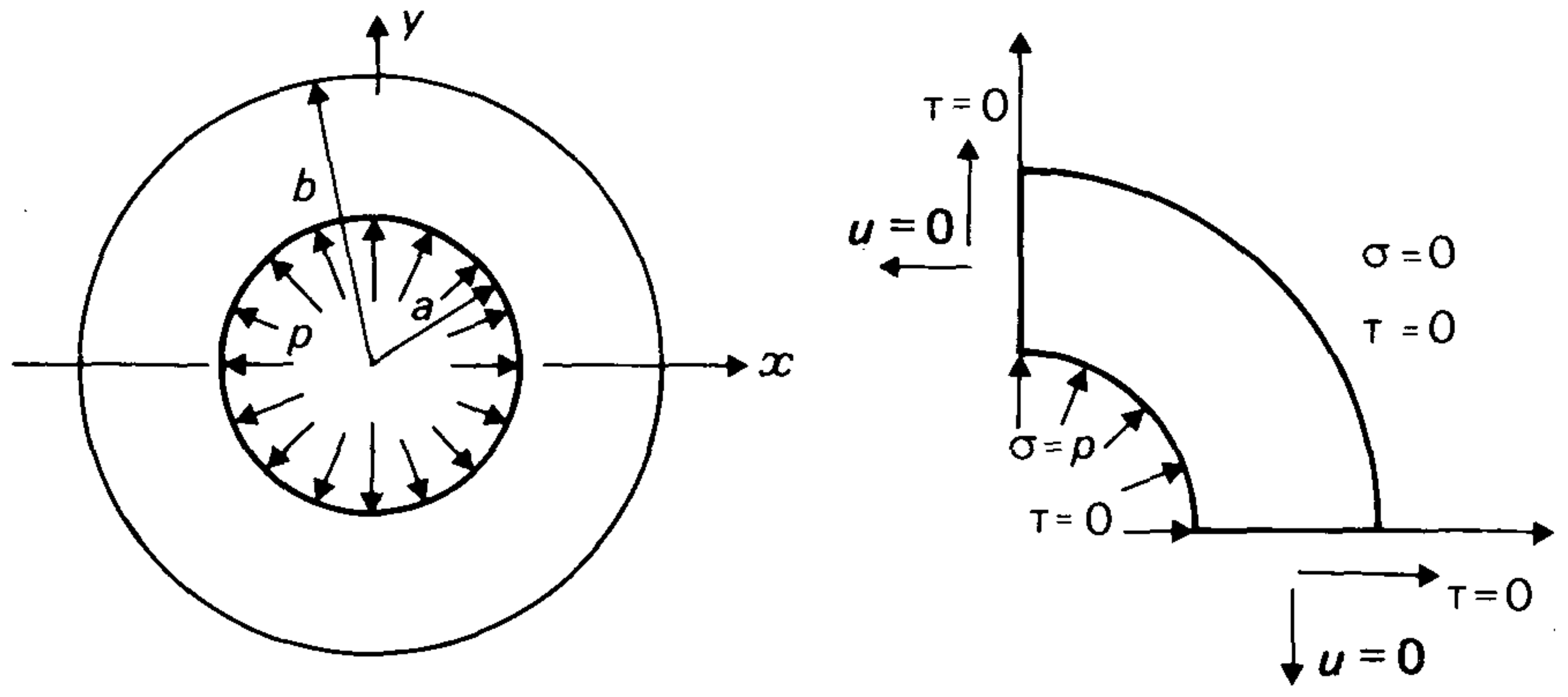

Figure 14. *Problem definition. Example* 3: *thick cylinder with internal pressure*, p = 1000 *kg/cm*^{2}; E = 2 × 10^5 *kg/cm*^{2}; $\nu = 0.25$; a = 10; b = 25

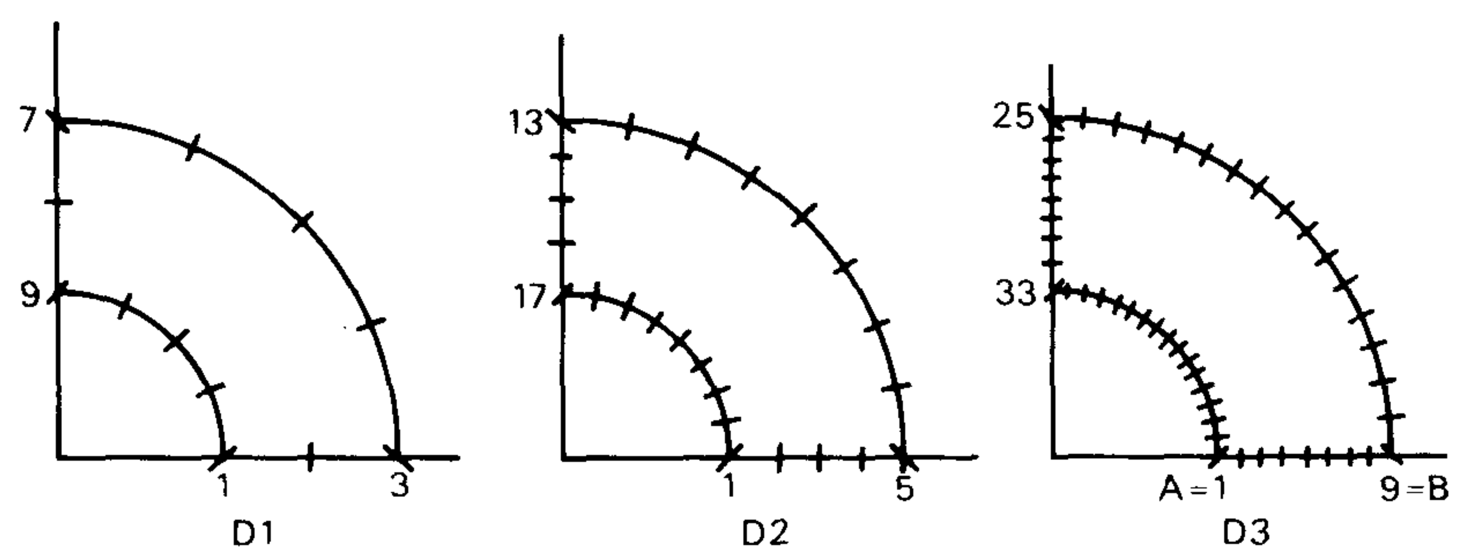

Figure 15. *Discretizations used with B.I.E.M.*

where (for instance):

$$A_{11}(k, x) = \int_{\partial D_k} T_{11}(k, x) N_1 \mathrm{d}s_k$$

$$B_{12}(k, x) = \int_{\partial D_k} U_{11}(k, x) N_2 \mathrm{d}s_k$$

These integration constants A and B, are evaluated numerically when the elements over which one integrates (k) do not include the point where the integral equation is applied (x).

Otherwise the integration is carried out analytically. Non-revertible singularities which appear in analytical integrations are eliminated during the assemblage of the matrix of coefficients of the system of integral equations. Furthermore these integrations should be taken in the sense of Cauchy principal values.

Application of the boundary conditions allows us to write equation (16) in the form:

$$\mathbf{F} = \mathbf{K} \cdot \mathbf{X} \tag{17}$$

The fact that there are 6 unknowns at each node to begin with (u, v, σ_B, σ_A, τ_B, τ_A) and only two integral equations makes it impossible to apply the boundary conditions directly in certain cases. In particular, when displacements are specified (4 unknowns) it is necessary to make an additional assumption which is established in the following way:

Let us assume that in the neighbourhood of node k, displacements vary in a linear way over the domain which allows us to find the strain tensor ε_{ij} and the principal strains ε_{I}, $\varepsilon_{\mathrm{II}}$ as well as their principal directions v_{I}, v_{II}. Therefore stresses at both sides of the node, can be expressed as functions of the principal stresses:

$$\sigma_k(k) = a\sigma_{\mathrm{I}}(k) + b\sigma_{\mathrm{II}}(k)$$

$$\sigma_{k-1}(k) = c\sigma_{\mathrm{I}}(k) + d\sigma_{\mathrm{II}}(k)$$

and similarly for τ, which means that only two unknowns remain associated with the node (k): σ_{I} and σ_{II}.

Finally in order to avoid ill-conditioning of the matrix **K**, the integral equations are scaled by multiplying the B coefficients by the elasticity modulus and dividing by the maximum dimension of the domain under study. Displacement data are also divided by this length and stress data by the elasticity modulus.

FLOW-CHART

SERBA is divided into a main program and four subroutines. The flow-chart of the main program is shown below.

SUBROUTINE LOLE: Calculates the integration constants of the boundary fundamental equation. The constants that form the matrix which is multiplied by the stress vector are calculated in the element oriented coordinate system. In this way, the stress vector is expressed in the element oriented coordinate system.

SUBROUTINE SOLUG: Solves the equation system by Gauss method.

SUBROUTINE CALA: Solves the stress discontinuity problem that appears in sharp corners. Finds the values that relate the stress vector before and after the corner with the principal stress vector in the corner.

SUBROUTINE PIYAY: Calculates the integration constants over the elements, needed to obtain the stress values at the internal points.

INPUT DATA CARDS

The input data cards are classified in the following way:

Two cards describe the general features of the problem.
Block of cards with internal points coordinates.
Block of cards with boundary nodes coordinates.
Block of cards with boundary conditions (stress and/or displacements)
Card with support conditions, if necessary.

1. *Card* (3*I*5, 10 ×, 25*A*2)

Cols 1 to 5: NP = Total number of boundary elements

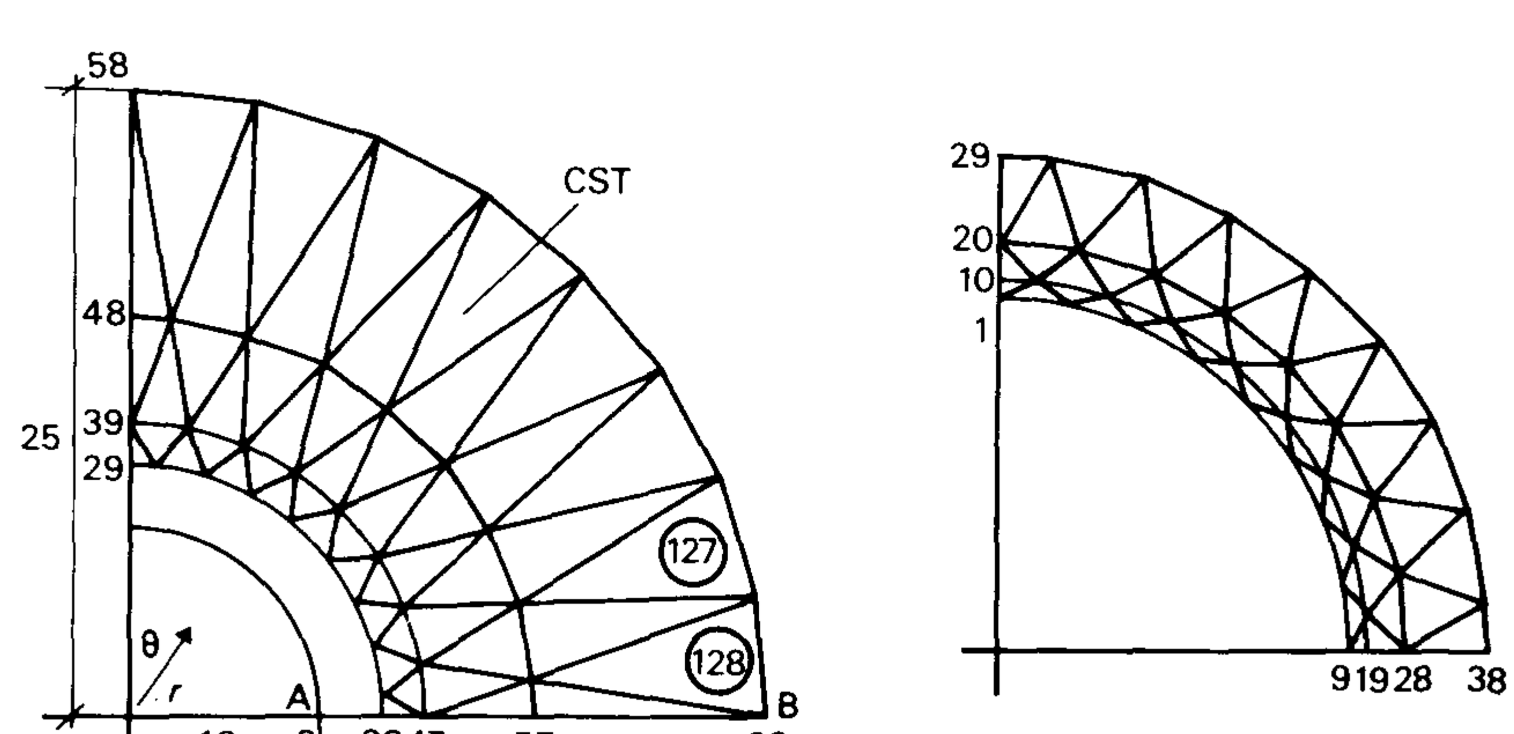

Figure 16. *Discretizations using F.E.M. Right: discretization in the internal region* (*example* 8)

Cols 6 to 10: NPI = Number of internal points
Cols 10 to 15: IANUL = Total number of restrained displacements (support conditions).
Cols 26 to 75: IENC = Title

2. *Card* (I5, 3F10.0)
Cols 1 to 5: NTIP = Problem type
1 for plane stress
0 for plane strain
Cols 6 to 15: E = Young's modulus
Cols 16 to 25: POIS = Poisson's ratio
Cols 26 to 35: DM = Longest dimension of the body (scaling of equations system).

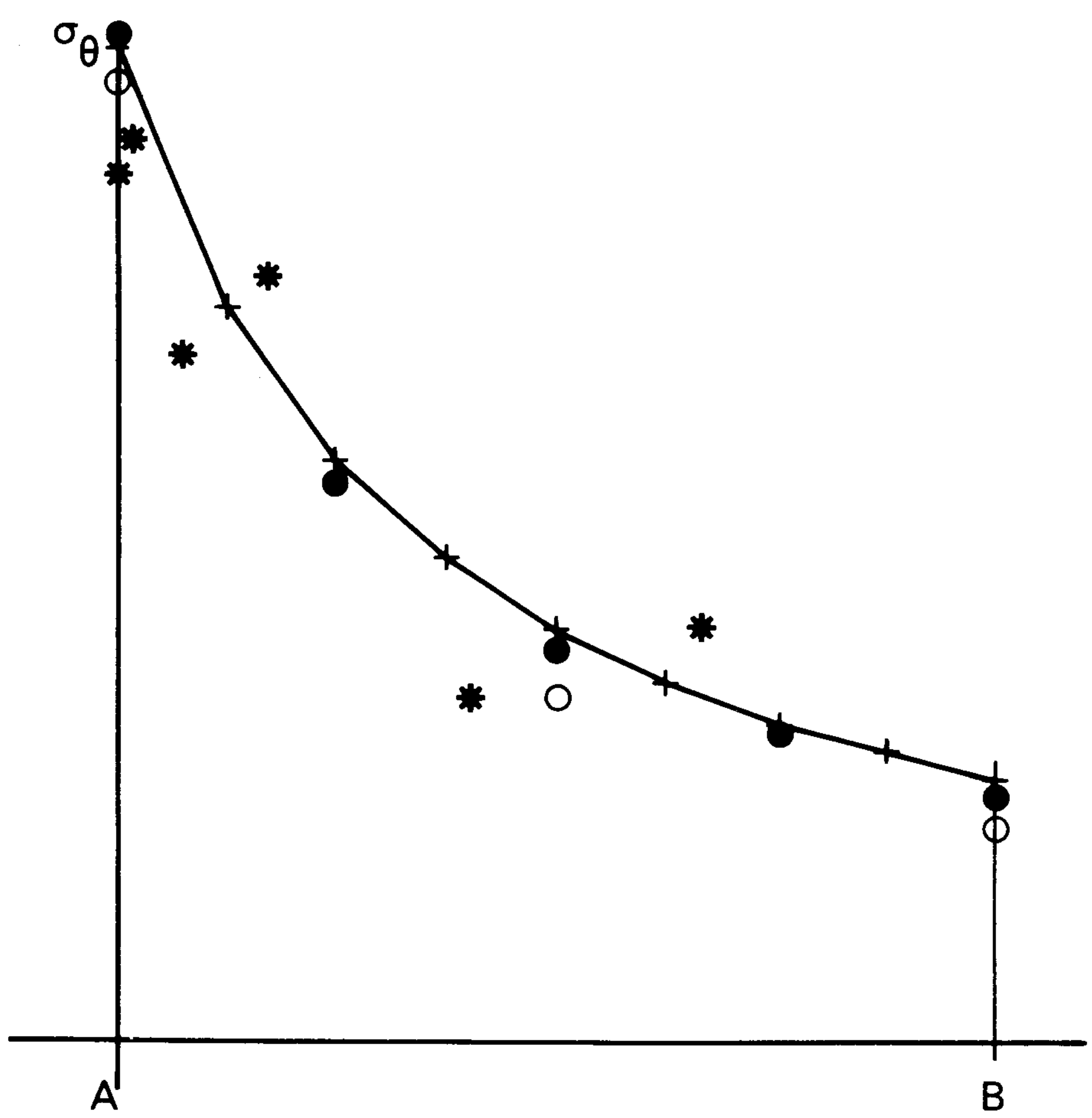

Figure 17. *Evolution of* σ_θ *over* A-B, ○, *B.I.E.M. with* 12 *elements;* ●, *B.I.E.M. with* 24 *elements;* +, *B.I.E.M. with* 36 *elements;* *, *F.E.M. with* 148 *elements* (*example* 3)

First block of cards
Internal points coordinates. Number of cards in this block is specified in the second variable of card 1. (2F10.0)
Cols 1 to 10: X coordinate of the internal point
Cols 11 to 20: Y coordinate of the internal point

Second block of cards
Boundary points, coordinates. (I5,2F10.0)
Cols 1 to 5: Nodal point number
Cols 6 to 15: X coordinate of the nodal point number
Cols 16 to 25: Y coordinate of the nodal point number
If the nodal point numbers are not consecutive, the program calculates the coordinates of the nodes in between, in a straight line, uniformly distributed.

Third block of cards
Boundary conditions. (I5,4(I5,E10.3))
Cols 1 to 5: Nodal point number
Cols 6 to 10: Integer numbers that identify the real variable
Cols 21 to 25: that comes next. They have to be ordered.
Cols 36 to 40:
Cols 51 to 55:
Cols 11 to 20: Values of the stress or displacement iden-
Cols 26 to 35: tified by the preceding integer number,

```
 1         24    3                            THICK CYLINDER WITH INTERNAL PRESSURE
 2             200000.   0.25                  15.
 3       10.           10.
 4       12.5          12.5
 5       15.           15.
 6          1 10.
 7          5 25.
 8          6 24.51963  4.87726
 9          7 23.09699  9.56709
10          8 20.78674 13.88926
11          9 17.67767 17.67767
12         10 13.88926 20.78674
13         11 9.56709  23.09699
14         12 4.87726  24.51963
15         13          25.
16         17          10.
17         181.95090   9.80785
18         193.82683   9.2388
19         205.55570   8.3147
20         217.07107   7.07107
21         228.3147    5.5557
22         239.2388    3.82683
23         249.80785   1.9509
24          1    3                   5 100.         6              8
25          5    1                   6              7              8
26         13    3                   5              6              8
27         17    1                   6              7 100.         8
28         25    3                   5 100.         6              8
```

Figure 18. *Input data cards for example* 3

Table 1

	Integer variable						Integer variable				
Case	I1	I2	I3	I4	Code	Case	I1	I2	I3	I4	Code
M-M(Sm)	1	2			1230	MT-T	1	6	7	8	1456
M-M(Sh)	1	2	3	4	1200		2	5	7	8	1356
T-T	5	6	7	8	3456		3	5	6	8	1346
MT-MT (Sh)	1	3	6	8	3681		4	5	6	7	1345
	2	4	5	7	4572	M-T	1	2	7	8	5001
	1	4	6	7	4671		3	4	5	6	5002
	2	3	5	8	3582	M-MT	1	2	3	8	5500
MT-MT (Sm)	1	6			2400		1	2	4	7	5501
	2	5			2300	MT-M	1	3	4	6	5502
							2	3	4	5	5503

MT-M means that you know a component of displacements and another of stress in the previous element and the two components of displacement in the following element.
Sm = Smooth boundary
Sh = Sharp corner

****
PROGRAM SERBA FOR SOLUTION OF TWO DIMENSIONALELASTICITY PROBLEMS USING B.I.E.M.
****

INPUT DATA

THICK CYLINDER WITH INTERNAL PRESSURE

NUMBER OF ELEMENTS = 24
NUMBER OF INTERNAL POINTS = 3
ELASTICITY MODULUS = 200000.0
POISSON RATIO =.250
PROBLEM DEFINITION = 0
1 = PLANE STRESS
0 = PLANE STRAIN

NODE	COOR X	COOR Y	CODE	U	V	UG	VG	SGB	TAUB	SGA	TAUA
1	10.0000	.0000	1346	.00000	.00000	.00000	.00000	.1000+003	.0000	.0000	.0000
2	13.7500	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
3	17.5000	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
4	21.2500	.0000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
5	25.0000	.0000	1456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
6	24.5196	4.8773	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
7	23.0970	9.5671	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
8	20.7867	13.8893	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
9	17.6777	17.6777	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
10	13.8893	20.7867	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
11	9.5671	23.0970	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
12	4.8773	24.5196	3456	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
13	.0000	25.0000	1346	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
14	.0000	21.2500	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
15	.0000	17.5000	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
16	.0000	13.7500	2400	.00000	.00000	.00000	.00000	.0000	.0000	.0000	.0000
17	.0000	10.0000	1456	.00000	.00000	.00000	.00000	.0000	.0000	.1000+003	.0000
18	1.9509	9.8079	3456	.00000	.00000	.00000	.00000	.1000+003	.0000	.1000+003	.0000
19	3.8268	9.2388	3456	.00000	.00000	.00000	.00000	.1000+003	.0000	.1000+003	.0000
20	5.5557	8.3147	3456	.00000	.00000	.00000	.00000	.1000+003	.0000	.1000+003	.0000
21	7.0711	7.0711	3456	.00000	.00000	.00000	.00000	.1000+003	.0000	.1000+003	.0000
22	8.3147	5.5557	3456	.00000	.00000	.00000	.00000	.1000+003	.0000	.1000+003	.0000
23	9.2388	3.8268	3456	.00000	.00000	.00000	.00000	.1000+003	.0000	.1000+003	.0000
24	9.8079	1.9509	3456	.00000	.00000	.00000	.00000	.1000+003	.0000	.1000+003	.0000
25	10.0000	.0000	1346	.00000	.00000	.00000	.00000	.1000+003	.0000	.0000	.0000

****

BOUNDARY RESULTS

NODE	XDISPLACEMENT	YDISPLACEMENT	SIGMA BEFORE	SIGMA AFTER	TAU BEFORE	TAU AFTER
1	-.7915200-002	.0000000	.1000000+003	-.1384223+003	.0000000	.0000000
2	-.6131101-002	.0000000	-.7836876+002	-.7836876+002	.0000000	.0000000
3	-.5202283-002	.0000000	-.5652406+002	-.5652406+002	.0000000	.0000000
4	-.4680419-002	.0000000	-.4509790+002	-.4509790+002	.0000000	.0000000
5	-.4391283-002	.0000000	-.3470266+002	.0000000	.0000000	.0000000
6	-.4310326-002	-.8522678-003	.0000000	.0000000	.0000000	.0000000
7	-.4050711-002	-.1676291-002	.0000000	.0000000	.0000000	.0000000
8	-.3636821-002	-.2429998-002	.0000000	.0000000	.0000000	.0000000
9	-.3090082-002	-.3090082-002	.0000000	.0000000	.0000000	.0000000
10	-.2429999-002	-.3636821-002	.0000000	.0000000	.0000000	.0000000
11	-.1676290-002	-.4050711-002	.0000000	.0000000	.0000000	.0000000
12	-.8522686-003	-.4310326-002	.0000000	.0000000	.0000000	.0000000
13	.0000000	-.4391281-002	.0000000	-.3470267+002	.0000000	.0000000
14	.0000000	-.4680418-002	-.4509799+002	-.4509799+002	.0000000	.0000000
15	.0000000	-.5202283-002	-.5652403+002	-.5652403+002	.0000000	.0000000
16	.0000000	-.6131100-002	-.7836872+002	-.7836872+002	.0000000	.0000000
17	.0000000	-.7915199-002	-.1384224+003	.1000000+003	.0000000	.0000000
18	-.1480788-002	-.7789527-002	.1000000+003	.1000000+003	.0000000	.0000000
19	-.2974641-002	-.7310800-002	.1000000+003	.1000000+003	.0000000	.0000000
20	-.4356931-002	-.6564153-002	.1000000+003	.1000000+003	.0000000	.0000000
21	-.5568152-002	-.5568152-002	.1000000+003	.1000000+003	.0000000	.0000000
22	-.6564154-002	-.4356931-002	.1000000+003	.1000000+003	.0000000	.0000000
23	-.7310800-002	-.2974641-002	.1000000+003	.1000000+003	.0000000	.0000000
24	-.7789527-002	-.1480787-002	.1000000+003	.1000000+003	.0000000	.0000000

INTERNAL POINTS RESULTS

COOR X	COOR Y	XDISPLACEMENT	YDISPLACEMENT	SIGMA X	SIGMA Y	TAU XY
10.000	10.000	-.42171247-002	-.42171236-002	-.19144268+002	-.19144264+002	.58653156+002
12.500	12.500	-.36338496-002	-.36338485-002	-.18907957+002	-.18907960+002	.37408140+002
15.000	15.000	-.32944230-002	-.32944225-002	-.18764558+002	-.18764569+002	.25946915+002

Figure 19. Results for example 3

Cols 41 to 50: according to the relations listed below.
Cols 56 to 65:
If there are no discontinuities in geometry, stress and displacements, only two integer number and two real variables need to be given.

List that identifies the variables in the third block

Integer value	*Real variable associated*
1	Normal displacement before u_B
2	Tangential displacement before v_B
3	Normal displacement after u_A
4	Tangential displacement after v_A
5	Normal stress before σ_B
6	Tangential stress before τ_B
7	Normal stress after σ_A
8	Tangential stress after τ_A

For the internal use of the program and for the data-check there is a simplified code according to the relation list (Table 1).

Last card
Introduction of support condition. Use this card only when the third variable of the first card is not zero. Write as integer values as specified for this variable (less than five). (515)

Cols 1 to 5: To punch n means to fix the global
Cols 6 to 10: displacement X in node n. To punch $n+N$
Cols 11 to 15: (N = total node number) means to fix the
Cols 16 to 20: global displacement Y in node n. If a local
Cols 21 to 25: displacement is known it is only possible to fix the global displacement Y.

GENERAL REMARKS

The nodes are numbered in an anticlockwise direction.

The cards specifying nodal characteristics (geometric or boundary conditions) should be ordered (when there is automatic generation nodes are not numbered consecutively in the cards).

The integration over the elements is performed numerically using Gauss quadrature with four points, when the elements does not include the node under consideration. When the element includes the node under consideration the integration is performed analytically.

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